

NORMS OF INDECOMPOSABLE INTEGERS IN REAL QUADRATIC FIELDS

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ABSTRACT. We study totally positive, additively indecomposable integers in a real quadratic field $\mathbb{Q}(\sqrt{D})$. We estimate the size of the norm of an indecomposable integer by expressing it as a power series in u_i^{-1} , where \sqrt{D} has the periodic continued fraction expansion $[u_0, u_1, u_2, \dots, u_{s-1}, 2u_0]$. This enables us to disprove a conjecture of Jang-Kim [JK] concerning the maximal size of the norm of an indecomposable integer.

1. INTRODUCTION

A totally positive integer in a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ is (additively) indecomposable if it can't be expressed as the sum of two totally positive integers. Indecomposable integers can be explicitly described using the continued fraction $\sqrt{D} = [u_0, \overline{u_1, u_2, \dots, u_{s-1}, 2u_0}]$ and are deeply connected to the structure of the number field $\mathbb{Q}(\sqrt{D})$. Recently Blomer and the author [BK], [Ka] investigated the relation between indecomposable elements and universal quadratic forms over \mathcal{O}_K , and showed that if there are M indecomposables (satisfying certain additional properties), then every universal, totally positive definite, quadratic form over \mathcal{O}_K has at least M variables.

One of the tools was the easy observation [BK, Lemma 3] that every totally positive integer, which is not divisible by any rational integer and has sufficiently small norm (at most \sqrt{D}), is indecomposable. This can be viewed as a lower bound on the norm that guarantees indecomposability. On the other hand, there are only finitely many indecomposables up to multiplication by units, and so there is a maximum of their norms.

The search for such an upper bound on the norm was started by Dress and Scharlau [DS] in 1982, when they proved that every indecomposable integer has norm less than or equal to D . Their result was recently improved by Jang and Kim [JK], who showed that in fact the maximum is at most $\frac{D}{N}$, where N is the minimum of absolute values of negative norms of elements of \mathcal{O}_K . (Both of these results can be improved when $D \equiv 1 \pmod{4}$ and $\mathbb{Z}[\sqrt{D}] \neq \mathcal{O}_K$ – however, for simplicity we restrict only to the case $D \equiv 2, 3 \pmod{4}$ in this paper).

Jang and Kim also proved that the upper bound is optimal in some cases and stated a general conjecture concerning an improvement of the bound, which we repeat as Conjecture 3 below.

In this note, we show that the conjecture doesn't hold. As an illustration we first provide a specific counterexample in Theorem 4. Then we give algorithms for computing power series expansions for the norms of negative convergents and indecomposables, compute the first

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few terms explicitly (Theorem 8 and Proposition 11), and determine which indecomposable elements have large norms (Proposition 10). These results have guided us towards finding the example in Theorem 4, but one can also most likely use them to obtain infinitely many counterexamples, as we indicate at the end of the paper. This argument and the proof of Theorem 4 involve a fairly routine verification in Mathematica.

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2. CONJECTURE OF JANG-KIM

Throughout this paper, let D be a squarefree positive integer. We shall work in the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ and its ring of integers \mathcal{O}_K . For simplicity, we always assume that $D \equiv 2, 3 \pmod{4}$, so that $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$ (although the arguments can be easily modified to cover the case $D \equiv 1 \pmod{4}$ as well).

For $\alpha = x + y\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ we denote its conjugate by $\alpha' = x - y\sqrt{D}$ and its norm by $N(\alpha) = \alpha\alpha' = x^2 - Dy^2$. We write $\alpha \succ \beta$ to denote $\alpha > \beta$ and $\alpha' > \beta'$, and say that α is totally positive if $\alpha \succ 0$. An element $\alpha \in \mathcal{O}_K$ is (additively) indecomposable if there are no $\beta, \gamma \in \mathcal{O}_K$ such that $\alpha = \beta + \gamma$ and $\beta, \gamma \succ 0$.

We shall use the following notation and its well-known properties (see eg. [HW], [Pe] for a reference):

- $\sqrt{D} = [u_0, \overline{u_1, u_2, \dots, u_{s-1}, 2u_0}]$ is a periodic continued fraction with $D \equiv 2, 3 \pmod{4}$ a squarefree positive integer
- $s, u_0, u_1, u_2, \dots, u_{s-1} \in \mathbb{N}$, $u_{si} = 2u_0$, and $u_{si+j} = u_j$ for $i > 0$ and $j \geq 0$
- the sequence $(u_1, u_2, \dots, u_{s-1})$ is symmetric, i.e., $u_{s-i} = u_i$ for $1 \leq i \leq s-1$
- $\frac{p_i}{q_i} := [u_0, \dots, u_i]$ is the i th convergent to \sqrt{D}
- $p_{i+1} = u_{i+1}p_i + p_{i-1}$ and $q_{i+1} = u_{i+1}q_i + q_{i-1}$ (with initial conditions $p_{-1} = 1$, $p_0 = u_0$, $q_{-1} = 0$, $q_0 = 1$)
- $p_{i+1}q_i - p_iq_{i+1} = (-1)^i$
- $\alpha_i := p_i + q_i\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ for $i \geq -1$
- $\alpha_i \succ 0 \Leftrightarrow i$ is odd
- $N_i := |N(\alpha_i)| = |p_i^2 - Dq_i^2| = (-1)^{i+1}N(\alpha_i)$
- $N := \min \left\{ |N(\alpha)|, \alpha \in \mathbb{Z}[\sqrt{D}] \text{ such that } N(\alpha) < 0 \right\} = \min \{ |N(\alpha_i)|, i \text{ even} \}$ is the minimum of absolute values of negative norms
- $T_i := p_i p_{i-1} - D q_i q_{i-1}$
- $\alpha_i \alpha'_{i+1} = T_{i+1} + (-1)^i \sqrt{D}$
- $c_i := [u_i, u_{i+1}, u_{i+2}, \dots]$, $c_i = u_i + \frac{1}{c_{i+1}}$
- $\sqrt{D} = \frac{c_{i+1}p_i + p_{i-1}}{c_{i+1}q_i + q_{i-1}}$
- $\alpha_{i,r} := \alpha_i + r\alpha_{i+1}$ is a semiconvergent for $i \geq -1$ and $0 \leq r \leq u_{i+2}$
- $\alpha_{i,0} = \alpha_i$, $\alpha_{i,u_{i+2}} = \alpha_{i+2}$
- $\alpha_{i,r} \succ 0$ for $0 \leq r \leq u_{i+2} \Leftrightarrow i$ is odd
- $M_i := N(\alpha_{i, \lfloor u_{i+2}/2 \rfloor})$

It is well-known that all the indecomposable integers are (some of) the semiconvergents, see eg. [Pe, §16 Nebennäherungsbrüche].

Proposition 1. *The indecomposable integers of $\mathbb{Z}[\sqrt{D}]$ are exactly the semiconvergents $\alpha_{i,r}$, $\alpha'_{i,r}$ for odd $i \geq -1$ and $0 \leq r < u_{i+2}$.*

We are interested in estimating the maximal norm of an indecomposable integer. First of all, Jang and Kim proved the following interesting result.

Theorem 2. [JK, Theorem 5]

a) *Let i be odd. Then*

$$N(\alpha_{i,r}) = \frac{D - (T_{i+1} + rN(\alpha_{i+1}))^2}{|N(\alpha_{i+1})|}.$$

b) *If $\alpha \in \mathbb{Z}[\sqrt{D}]$ is indecomposable, then $N(\alpha) \leq \frac{D}{N}$.*

Proof. For the sake of completeness, we briefly indicate the proof, following [JK]. We have

$$\begin{aligned} N(\alpha_{i,r}) &= N(\alpha_i + r\alpha_{i+1}) = N\left(\frac{1}{\alpha'_{i+1}} \cdot \alpha'_{i+1}(\alpha_i + r\alpha_{i+1})\right) = \frac{N(\alpha'_{i+1}\alpha_i + rN(\alpha_{i+1}))}{N(\alpha_{i+1})} = \\ &= \frac{N(T_{i+1} + rN(\alpha_{i+1}) + (-1)^i\sqrt{D})}{N(\alpha_{i+1})} = \frac{(T_{i+1} + rN(\alpha_{i+1}))^2 - D}{N(\alpha_{i+1})} = \\ &= \frac{D - (T_{i+1} + rN(\alpha_{i+1}))^2}{|N(\alpha_{i+1})|} \leq \frac{D}{N}, \end{aligned}$$

since $|N(\alpha_{i+1})| \geq N$ by the definition of $-N$ as the maximum of negative norms of elements of $\mathbb{Z}[\sqrt{D}]$. Together with Proposition 1, this immediately implies b). \square

Let i_0 be an index such that the minimum N of absolute values of negative norms satisfies $N = N_{i_0+1} (= |N(\alpha_{i_0+1})|)$. Motivated by the preceding Theorem 2, Jang and Kim expected that the maximum norm of an indecomposable integer is attained at $\alpha_{i,r}$ for $i = i_0$ and some r . This expectation then led them to make the following conjecture.

Conjecture 3. [JK, Conjecture 1] *Let a be the smallest nonnegative rational integer such that N divides $D - a^2$. Then $N(\alpha) \leq \frac{D-a^2}{N}$ for all indecomposable $\alpha \in \mathbb{Z}[\sqrt{D}]$.*

However, the expectation need not be true, i.e., we can have $N(\alpha_{j,t}) > N(\alpha_{i_0,r})$ for some $j \neq i_0$, and then the conjecture may not hold.

Theorem 4. *Let $D = 24\,009\,857\,226\,825\,282\,345\,490$. Then:*

(1) $D \equiv 2 \pmod{4}$ is squarefree and its continued fraction is

$$\sqrt{D} = [u_0, \overline{10, 2, 12, 6, 1, 3, 4, 3, 12, 3, 4, 2, 1, 6, 12, 2, 10, 2u_0}]$$

with $u_0 = 154\,951\,144\,645$.

(2) α_2 has the largest negative norm $-N = -N_2 = -24\,548\,583\,881$

(3) $\alpha_{7,6}$ is the indecomposable integer with the largest norm $M_7 = 977\,608\,342\,706$

(4) The smallest nonnegative rational integer a such that N divides $D - a^2$ is $a = 4030\,160\,489$.

(5) $977\,608\,342\,706 = M_7 > \frac{D-a^2}{N} = M_1 = 977\,393\,040\,249$

Hence Conjecture 3 is false over $\mathbb{Q}(\sqrt{D})$.

Proof. These results are easily verified by a computation in Mathematica, the file with the computations is available at sites.google.com/site/vitakala/research/indec.

(1) is straightforward. For (2), we know that the element with largest negative norm is some α_i with even i , $0 \leq i \leq s = 18$. Hence we just need to check these 10 possibilities. (Note that the second largest is α_8 with norm $-N_8 = -24\,559\,791\,665$.)

By Proposition 1 we know that (up to multiplication by units and conjugation), $\alpha_{i,r}$ with odd i , $-1 \leq i \leq s-2$ and $0 \leq r < u_{i+2}$, are exactly the indecomposable integers. Again this is a small set of values that we need to check to prove (3) (in fact, we could restrict it even more using Proposition 10).

(4) is obtained by solving the congruence $a^2 \equiv D \pmod{N}$, and (5) then follows. \square

Our goal in the rest of the paper is to give good estimates on the sizes of N_i and $N(\alpha_{i,r})$, to use them to explain how we found the counterexample in Theorem 4, and then to outline the argument that there are most likely infinitely many of them.

Remark. Note that the assumption that α is indecomposable is missing from the statement of Conjecture 1 in [JK]. Also, in the discussion immediately preceding the conjecture, there should probably be “ $x + tN(p_{s-1}) = a$ ”.

3. NORMS OF CONVERGENTS

In this section we give an algorithm for expressing $N_i := |N(\alpha_i)|$ as a power series in $u_0^{-1}, u_1^{-1}, u_2^{-1}, \dots, u_{s-1}^{-1}$ and compute the first few terms (Theorem 8). For this purpose, we first prove recurrence relations for $1/c_i$ and N_i .

Proposition 5. *For $i \geq 0$ we have*

a)

$$\frac{1}{c_i} = \frac{1}{u_i} \left(1 - \frac{1}{u_i c_{i+1}} + \frac{1}{u_i^2 c_{i+1}^2} - \dots \right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{u_i^{j+1} c_{i+1}^j},$$

b)

$$T_i = (-1)^{i+1} \sqrt{D} - N(\alpha_i) c_{i+1},$$

c)

$$N_i = \frac{2\sqrt{D}}{c_{i+1}} - \frac{N_{i-1}}{c_{i+1}^2},$$

and

d)

$$\frac{N_i}{c_{i+2}} < \sqrt{D}.$$

Proof. From the definition of c_i we have

$$c_i = [u_i, u_{i+1}, u_{i+2}, \dots] = u_i + \frac{1}{[u_{i+1}, u_{i+2}, \dots]} = u_i + \frac{1}{c_{i+1}}.$$

To prove a), we take the reciprocal of this formula and obtain

$$\frac{1}{c_i} = \frac{1}{u_i + \frac{1}{c_{i+1}}} = \frac{1}{u_i} \cdot \frac{1}{1 + \frac{1}{u_i c_{i+1}}} = \frac{1}{u_i} \cdot \sum_{j=0}^{\infty} (-1)^j \frac{1}{u_i^j c_{i+1}^j}.$$

Since $u_i c_{i+1} = u_i(u_{i+1} + \frac{1}{c_{i+2}}) \geq 1 \cdot (1 + \frac{1}{c_{i+2}}) > 1$, the series converges absolutely.

For the second formula b), note that since $\sqrt{D} = \frac{c_{i+1}p_i + p_{i-1}}{c_{i+1}q_i + q_{i-1}}$, we have $c_{i+1}(p_i - q_i\sqrt{D}) = -p_{i-1} + q_{i-1}\sqrt{D}$. This in turn implies

$$c_{i+1} = \frac{-p_{i-1} + q_{i-1}\sqrt{D}}{p_i - q_i\sqrt{D}} = \frac{(p_i + q_i\sqrt{D})(-p_{i-1} + q_{i-1}\sqrt{D})}{(p_i + q_i\sqrt{D})(p_i - q_i\sqrt{D})} = -\frac{T_i + (-1)^i\sqrt{D}}{N(\alpha_i)},$$

i.e., $T_i = (-1)^{i+1}\sqrt{D} - N(\alpha_i)c_{i+1}$.

To prove c), we can also express

$$T_{i+1} = p_{i+1}p_i - q_{i+1}q_iD = (u_{i+1}p_i + p_{i-1})p_i - (u_{i+1}q_i + q_{i-1})q_iD = u_{i+1}N(\alpha_i) + T_i.$$

Plugging in the expression for T_i above, we see that

$$T_{i+1} = (-1)^{i+1}\sqrt{D} - N(\alpha_i)(c_{i+1} - u_{i+1}) = (-1)^{i+1}\sqrt{D} - \frac{N(\alpha_i)}{c_{i+2}}.$$

Finally,

$$(-1)^{i+1}\sqrt{D} - N(\alpha_i)c_{i+1} = T_i = (-1)^i\sqrt{D} - \frac{N(\alpha_{i-1})}{c_{i+1}},$$

and hence

$$N(\alpha_i)c_{i+1} = 2 \cdot (-1)^{i+1}\sqrt{D} + \frac{N(\alpha_{i-1})}{c_{i+1}}.$$

This finishes the proof of c) using the definition $N_j = (-1)^{j+1}N(\alpha_j)$ for $j = i, i+1$.

d) Using c), we see that

$$\frac{N_i}{c_{i+2}} < \frac{2\sqrt{D}}{c_{i+1}c_{i+2}} = \frac{2\sqrt{D}}{(u_{i+1} + 1/c_{i+2})c_{i+2}} = \frac{2\sqrt{D}}{u_{i+1}c_{i+2} + 1} < \sqrt{D}.$$

□

Recursively using the formulas from Proposition 5, one can obtain the desired power series expression for N_i as Theorem 8. The only term of total degree 1 (in $u_0^{-1}, u_1^{-1}, u_2^{-1}, \dots, u_{s-1}^{-1}$) will be $\frac{2\sqrt{D}}{u_{i+1}}$, which corresponds to the well-known approximation $N_i \approx \frac{2\sqrt{D}}{u_{i+1}}$ (see eg. [Ka, Proposition 3.3]). In fact, we can improve this estimate as follows (it is an improvement, as $c_{i+1} = u_{i+1} + 1/c_{i+2} > u_{i+1}$).

Proposition 6. *For $i \geq 0$ we have*

$$\frac{2\sqrt{D}}{c_{i+1}} \left(1 - \frac{1}{c_i c_{i+1}}\right) < N_i < \frac{2\sqrt{D}}{c_{i+1}}.$$

Proof. The upper bound follows directly from Proposition 5c), as we have $N_i = \frac{2\sqrt{D}}{c_{i+1}} - \frac{N_{i-1}}{c_{i+1}^2} < \frac{2\sqrt{D}}{c_{i+1}}$.

To prove the lower bound, we apply the upper bound to Proposition 5c) again

$$N_i = \frac{2\sqrt{D}}{c_{i+1}} - \frac{N_{i-1}}{c_{i+1}^2} > \frac{2\sqrt{D}}{c_{i+1}} - \frac{2\sqrt{D}}{c_i} \cdot \frac{1}{c_{i+1}^2}.$$

□

To be able to estimate the error of the estimates, we first need to estimate the errors (coming from truncating the series) when applying the formulas from Proposition 5. This is routine, but somewhat technical.

Lemma 7. a) For every $k \geq 1$ there is $-1 \leq \varepsilon \leq 1$ such that

$$\frac{1}{c_i} = \sum_{j=0}^{k-1} \frac{(-1)^j}{u_i^{j+1} c_{i+1}^j} + \frac{\varepsilon}{u_i^{k+1} u_{i+1}^k}.$$

b) For some $-1 \leq \varepsilon \leq 1$ we have

$$\frac{1}{c_i} = \frac{1}{u_i} \left(1 - \frac{1}{u_i u_{i+1}} \right) + \frac{1}{u_i^2 u_{i+1}^2} \left(\frac{1}{u_i} + \frac{1}{u_{i+2}} \right) \varepsilon.$$

c) For some $-1 \leq \varepsilon \leq 1$ we have

$$\frac{1}{c_i^2} = \frac{1}{u_i^2} + \frac{2}{u_i^3 u_{i+1}} \cdot \varepsilon.$$

d) For some $-1 \leq \varepsilon \leq 1$ we have

$$\frac{1}{c_i^2} = \frac{1}{u_i^2} \left(1 - \frac{2}{u_i u_{i+1}} \right) + \frac{2}{u_i^4 u_{i+1}^2} \cdot \varepsilon.$$

Proof. a) By Proposition 5, we have

$$\frac{1}{c_i} = \sum_{j=0}^{k-1} \frac{(-1)^j}{u_i^{j+1} c_{i+1}^j} \pm \frac{1}{u_i^{k+1} c_{i+1}^k} \cdot \sum_{j=0}^{\infty} \frac{(-1)^j}{u_i^j c_{i+1}^j} = \sum_{j=0}^{k-1} \frac{(-1)^j}{u_i^{j+1} c_{i+1}^j} \pm \frac{1}{u_i^{k+1} c_{i+1}^k} \cdot \frac{1}{1 + \frac{1}{u_i c_{i+1}}},$$

and so the error satisfies

$$\left| \pm \frac{1}{u_i^{k+1} c_{i+1}^k} \cdot \frac{1}{1 + \frac{1}{u_i c_{i+1}}} \right| \leq \frac{1}{u_i^{k+1} u_{i+1}^k} \cdot 1.$$

b) follows from a) with $k = 2$ by estimating the term $1/c_{i+1}$ again using a) with $k = 1$.

c), d) We have $\frac{1}{c_i} = \frac{1}{u_i} \cdot \frac{1}{1 + \frac{1}{u_i c_{i+1}}}$, and so

$$\frac{1}{c_i^2} = \frac{1}{u_i^2} \cdot \frac{1}{1 + \frac{2}{u_i c_{i+1}} \left(1 + \frac{1}{2u_i c_{i+1}} \right)} = \frac{1}{u_i^2} \sum_{j=0}^{\infty} \frac{(-2)^j}{u_i^j c_{i+1}^j} \left(1 + \frac{1}{2u_i c_{i+1}} \right)^j.$$

From here c) and d) follow similarly as in the proof of part a). \square

We are finally ready to give the desired expansion for N_i . In the next section we shall need to consider the terms of degree at most 5. Especially when the coefficients u_j are not too small, this gives us very good information on the approximate size of N_i . On the other hand, when eg. $u_i = u_{i+1} = u_{i+2} = 1$, the formulas are nearly useless.

Theorem 8. a) Degree 1: For some $0 < \varepsilon \leq 1$ we have

$$\frac{N_i}{2\sqrt{D}} = \frac{1}{u_{i+1}} - \frac{1}{u_{i+1}^2} \left(\frac{1}{u_i} + \frac{1}{u_{i+2}} \right) \varepsilon.$$

b) Degree 3: Assume that $u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3} \geq u$ for some $u \in \mathbb{N}$. Then there is some $-1 \leq \varepsilon \leq 1$ such that

$$\frac{N_i}{2\sqrt{D}} = \frac{1}{u_{i+1}} \left(1 - \frac{1}{u_i u_{i+1}} - \frac{1}{u_{i+1} u_{i+2}} \right) + \frac{10}{u^5} \cdot \varepsilon.$$

c) Degree 5: We have

$$N_i = \frac{2\sqrt{D}}{u_{i+1}} \left[1 - \frac{1}{u_i u_{i+1}} - \frac{1}{u_{i+1} u_{i+2}} + \left(\frac{1}{u_i u_{i+1}} + \frac{1}{u_{i+1} u_{i+2}} \right)^2 + \frac{1}{u_{i-1} u_i^2 u_{i+1}} + \frac{1}{u_{i+1} u_{i+2}^2 u_{i+3}} \right] + \dots,$$

where \dots stands for terms of total degree greater than 5. If $u \in \mathbb{N}$ is such that $u_j \geq u$ for all j , then the error satisfies $|\dots| < \frac{65}{u^7}$.

Proof. This is a routine repeated application of the formulas from Propositions 5, 6, and estimates from Lemma 7, and so we only illustrate it by proving parts a) and b), i.e., by a computation till degree 3:

a) By Proposition 5c), Lemma 7a) for $k = 1$, and Proposition 6 we have:

$$\left| \frac{N_i}{2\sqrt{D}} - \frac{1}{u_{i+1}} \right| \leq \left| \frac{1}{c_{i+1}} - \frac{1}{u_{i+1}} \right| + \left| \frac{N_{i-1}}{2\sqrt{D}} \cdot \frac{1}{c_{i+1}^2} \right| \leq \frac{1}{u_{i+1}^2 u_{i+2}} + \frac{1}{u_i c_{i+1}^2} \leq \frac{1}{u_{i+1}^2 u_{i+2}} + \frac{1}{u_i u_{i+1}^2}.$$

By Proposition 6, we see that the error has to be negative.

b) We first estimate all the u_j in the error terms by u , and then repeatedly apply Proposition 5c), Lemma 7, and part a) of this theorem as follows:

$$\begin{aligned} \frac{N_i}{2\sqrt{D}} &= \frac{1}{c_{i+1}} - \frac{N_{i-1}}{2\sqrt{D}} \cdot \frac{1}{c_{i+1}^2} = \\ &= \frac{1}{u_{i+1}} \left(1 - \frac{1}{u_{i+1} u_{i+2}} \right) + \frac{2\varepsilon_1}{u^5} + \left(\frac{1}{u_i} + \frac{2\varepsilon_2}{u^3} \right) \cdot \left(\frac{1}{u_{i+1}^2} + \frac{2\varepsilon_3}{u^4} \right) = \\ &= \frac{1}{u_{i+1}} \left(1 - \frac{1}{u_i u_{i+1}} - \frac{1}{u_{i+1} u_{i+2}} \right) + \frac{2\varepsilon_1}{u^5} + \frac{1}{u_i} \cdot \frac{2\varepsilon_3}{u^4} + \frac{2\varepsilon_2}{u^3} \cdot \frac{1}{u_{i+1}^2} + \frac{2\varepsilon_2}{u^3} \cdot \frac{2\varepsilon_3}{u^4}, \end{aligned}$$

and we see that the absolute value of the error is less than $\frac{2}{u^5} + \frac{2}{u^5} + \frac{2}{u^5} + \frac{4}{u^7} \leq \frac{10}{u^5}$.

The proof of c) is similar, only more technical. \square

Note that we can also use these results to give a simple proof of Proposition 1 from [JK], which says that there is an element of norm $\frac{D-a^2}{N}$ for some $a \in \mathbb{Z}$, $|a| \leq N/2$. Jang-Kim study the prime factorization of α_{i+1} to prove this, but it follows directly by combining Theorem 2 with Proposition 5.

Proposition 9. [JK, Proposition 1] *Let i be such that $N_{i+1} = N$. Then there is some $0 \leq r \leq u_{i+2}$ such that $N(\alpha_{i,r}) = \frac{D-a^2}{N}$ for some $a \in \mathbb{Z}$, $|a| \leq N/2$.*

Proof. Clearly i is odd, and by Theorem 2 we know that

$$N(\alpha_{i,r}) = \frac{D - (T_{i+1} - rN)^2}{N}.$$

If we take r_1 to be the integer for which $|T_{i+1} - rN|$ is minimal, then $|T_{i+1} - r_1 N| \leq N/2$ as we want. So we only need to check that such r_1 lies in the specified interval $0 \leq r \leq u_{i+2}$. This will follow if we show that $T_{i+1} - 0 \cdot N > 0 > T_{i+1} - u_{i+2}N$, which is easy to see using Proposition 5:

$r = 0$: We have

$$T_{i+1} - 0 \cdot N = -\sqrt{D} + N_{i+1} c_{i+2} = \sqrt{D} - \frac{N_i}{c_{i+2}} > 0,$$

where we first used 5b), then 5c), and finally 5d).

$r = u_{i+2}$: Similarly, we have

$$T_{i+1} - u_{i+2}N = -\sqrt{D} + N_{i+1}(c_{i+2} - u_{i+2}) = -\sqrt{D} + \frac{N_{i+1}}{c_{i+3}} < 0.$$

This finishes the proof (in fact, we shall see in Proposition 10 that $\frac{u_{i+2}}{2} - 1 < r_1 < \frac{u_{i+2}}{2} + 1$). \square

4. NORMS OF INDECOMPOSABLE ELEMENTS

In this section we determine for which value of r the indecomposable element $\alpha_{i,r}$ (with i fixed) has maximal norm (Proposition 10). Then we prove a power series formula for this norm, similar to Theorem 8. This will give us a heuristic for finding counterexamples to the Conjecture 3 of Jang-Kim.

Assume that i is odd so that $\alpha_{i,r} \succ 0$ is indecomposable for all $0 \leq r \leq u_{i+2}$. Let's first determine which value of r maximizes the norm of $\alpha_{i,r}$.

Proposition 10. *Assume that i is odd and let r_0 be such that $N(\alpha_{i,r})$ is maximal among $0 \leq r \leq u_{i+2}$. Then $\frac{u_{i+2}}{2} - 1 < r_0 < \frac{u_{i+2}}{2} + 1$.*

If u_{i+2} is even, then

$$r_0 = \frac{u_{i+2}}{2}.$$

Proof. By Theorem 2a) we have

$$N(\alpha_{i,r}) = \frac{D - (T_{i+1} + rN(\alpha_{i+1}))^2}{|N(\alpha_{i+1})|} = \frac{D - (T_{i+1} - rN_{i+1})^2}{N_{i+1}},$$

and so the norm is maximal when $|T_{i+1} - rN_{i+1}|$ is minimal, which happens when $\left| \frac{T_{i+1}}{N_{i+1}} - r \right|$ is minimal (with $0 \leq r \leq u_{i+2}$).

Let's start by showing that

$$(*) \quad \left| \frac{T_{i+1}}{N_{i+1}} - \frac{u_{i+2}}{2} \right| < \frac{1}{2}.$$

By Proposition 5b) we have $T_{i+1} = -\sqrt{D} + N_{i+1}c_{i+2}$, and hence $(*)$ is equivalent to

$$N_{i+1} > \left| -2\sqrt{D} + N_{i+1}(2c_{i+2} - u_{i+2}) \right| = \left| -2\sqrt{D} + N_{i+1} \left(c_{i+2} + \frac{1}{c_{i+3}} \right) \right|.$$

For this, we prove two inequalities:

$$\text{a) } N_{i+1} > -2\sqrt{D} + N_{i+1} \left(c_{i+2} + \frac{1}{c_{i+3}} \right):$$

By Proposition 6, we have

$$2\sqrt{D} > N_{i+1}c_{i+2} > N_{i+1} \left(c_{i+2} + \frac{1}{c_{i+3}} - 1 \right),$$

as we wanted to prove.

$$\text{b) } N_{i+1} > 2\sqrt{D} - N_{i+1} \left(c_{i+2} + \frac{1}{c_{i+3}} \right):$$

First note that

$$c_{i+1}c_{i+2} = \left(u_{i+1} + \frac{1}{c_{i+2}} \right) c_{i+2} = u_{i+1}c_{i+2} + 1 \geq c_{i+2} + 1,$$

and so

$$-\frac{1}{c_{i+1}c_{i+2}} + \frac{1}{c_{i+2}} - \frac{1}{c_{i+1}c_{i+2}^2} \geq 0.$$

Hence

$$\begin{aligned} 2\sqrt{D} &\leq 2\sqrt{D} - \frac{2\sqrt{D}}{c_{i+1}c_{i+2}} + \frac{2\sqrt{D}}{c_{i+2}} - \frac{2\sqrt{D}}{c_{i+1}c_{i+2}^2} = \frac{2\sqrt{D}}{c_{i+2}} \left(1 - \frac{1}{c_{i+1}c_{i+2}}\right) (c_{i+2} + 1) < \\ &< N_{i+1}(c_{i+2} + 1) < N_{i+1} \left(c_{i+2} + 1 + \frac{1}{c_{i+3}}\right), \end{aligned}$$

as we wanted to show (note that in the penultimate inequality we used Proposition 6). This proves (*).

Let $r_1 \in \mathbb{Z}$ be such that $\left|\frac{T_{i+1}}{N_{i+1}} - r_1\right|$ is minimal. Then $\left|\frac{T_{i+1}}{N_{i+1}} - r_1\right| \leq \frac{1}{2}$, and so using the triangle inequality and (*),

$$\left|r_1 - \frac{u_{i+2}}{2}\right| \leq \left|r_1 - \frac{T_{i+1}}{N_{i+1}}\right| + \left|\frac{T_{i+1}}{N_{i+1}} - \frac{u_{i+2}}{2}\right| < \frac{1}{2} + \frac{1}{2} = 1.$$

As r_1 is an integer and $u_{i+2} \geq 1$, we see that $0 \leq r_1 \leq u_{i+2}$, and so $r_0 = r_1$ and the proposition is proved. \square

From now on, we shall assume that i is odd and u_{i+2} even as in Proposition 10. Recall that we have defined $M_i := N(\alpha_{i,u_{i+2}/2})$. Let's now find a power series expression for the norm M_i till degree 3, which we shall then use to find the example of Theorem 4. (A similar formula holds also in the case of odd u_{i+2} , or even for arbitrary r , but it's more complicated.)

Proposition 11. *Let i be odd and u_{i+2} even. Then*

a) Degree 1:

$$\frac{2M_i}{\sqrt{D}} = u_{i+2} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} + \dots,$$

where \dots stands for terms of total degree in u_j^{-1} greater than 2.

b) Degree 3:

$$\frac{2M_i}{\sqrt{D}} = u_{i+2} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} - \left(\frac{1}{u_i u_{i+1}^2} + \frac{1}{u_{i+3}^2 u_{i+4}} + \frac{1}{u_{i+2}} \left(\frac{1}{u_{i+1}} - \frac{1}{u_{i+3}} \right)^2 \right) + \dots,$$

where \dots stands for terms of total degree in u_j^{-1} greater than 4. If $u \in \mathbb{N}$ is such that $u_j \geq u$ for all j , then the error satisfies $|\dots| < \frac{105}{u^5}$.

Proof. First let $0 \leq r \leq u_{i+2}$. We have

$$\begin{aligned} N(\alpha_{i,r}) &= (\alpha_i + r\alpha_{i+1})(\alpha'_i + r\alpha'_{i+1}) = N(\alpha_i) + r^2 N(\alpha_{i+1}) + 2r(p_i p_{i+1} - Dq_i q_{i+1}) \\ &= N_i - r^2 N_{i+1} + 2r T_{i+1}. \end{aligned}$$

If we set $r = u_{i+2}$, the left hand side becomes N_{i+2} and we have

$$2u_{i+2} T_{i+1} = N_{i+2} + u_{i+2}^2 N_{i+1} - N_i.$$

Let us now take $r = u_{i+2}/2$ and combine the preceding two formulas to get

$$\begin{aligned} 4M_i &= 4N_i - u_{i+2}^2 N_{i+1} + 2(N_{i+2} + u_{i+2}^2 N_{i+1} - N_i) \\ &= u_{i+2}^2 N_{i+1} + 2(N_{i+2} + N_i). \end{aligned}$$

Now we divide this equation by $2\sqrt{D}$ and apply the formulas from Theorem 8.

a)

$$\frac{2M_i}{\sqrt{D}} = u_{i+2}^2 \cdot \frac{1}{u_{i+2}} \left(1 - \frac{1}{u_{i+1}u_{i+2}} - \frac{1}{u_{i+2}u_{i+3}} \right) + 2 \left(\frac{1}{u_{i+3}} + \frac{1}{u_{i+1}} \right) + \dots,$$

which simplifies to the desired formula.

b) is similar, we just use degree 5 expansion for N_{i+1} and degree 3 for N_i and N_{i+2} . \square

Now we are ready to explain how we constructed the counterexample to Conjecture 3 in Theorem 4. The conjecture is based on the expectation that, as i varies, $N(\alpha_{i,r})$ is maximal for i such that $N_{i+1} = |N(\alpha_{i+1})|$ is minimal. Thus the key is to find some D and odd indices $i < j$ such that

- α_{i+1} is the element with the largest negative norm (= the smallest norm in absolute value $N_{i+1} = N$),
- $N_{i+1} = |N(\alpha_{i+1})| < N_{j+1} = |N(\alpha_{j+1})|$, but the difference of the norms is small,
- $M_i = N(\alpha_{i,r}) < M_j = N(\alpha_{j,t})$ for $r = u_{i+2}/2, t = u_{j+2}/2$ as in Proposition 10.

We shall do this by prescribing some of the coefficients u_i of the continued fraction for \sqrt{D} and using Friesen's theorem [Fr] that guarantees the existence of infinitely many such squarefree integers D . Since we are using only heuristics and not precise estimates, we then have to verify that all the conditions are indeed satisfied. Hence in the following discussion we ignore error terms (and place quotation marks around claims that are imprecise).

First of all, if the length s of period of the continued fraction for \sqrt{D} is odd, then the fundamental unit has the largest negative norm -1 . To avoid this situation we take s even.

From Theorem 8a), we see that $N_{i+1} < N_{j+1}$ “if and only if” $u_{i+2} \geq u_{j+2}$. But if $u_{i+2} > u_{j+2}$, then Proposition 11a) “implies” that $M_i > M_j$, which we don't want. Hence let's take $u_{i+2} = u_{j+2}$ and consider the higher order terms.

Theorem 8b) then says that $N_{i+1} < N_{j+1}$ “if and only if” $\frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} \geq \frac{1}{u_{j+1}} + \frac{1}{u_{j+3}}$. But again Proposition 11a) “implies” that if $M_i < M_j$, then strict inequality cannot occur, so that we have

$$\frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} = \frac{1}{u_{j+1}} + \frac{1}{u_{j+3}}.$$

In this case 8c) gives us $N_{i+1} < N_{j+1}$ “if and only if”

$$\frac{1}{u_i u_{i+1}^2} + \frac{1}{u_{i+3}^2 u_{i+4}} \leq \frac{1}{u_j u_{j+1}^2} + \frac{1}{u_{j+3}^2 u_{j+4}},$$

and 11b) says $M_i < M_j$ “if and only if”

$$\frac{1}{u_i u_{i+1}^2} + \frac{1}{u_{i+3}^2 u_{i+4}} + \frac{1}{u_{i+2}} \left(\frac{1}{u_{i+1}} - \frac{1}{u_{i+3}} \right)^2 \geq \frac{1}{u_j u_{j+1}^2} + \frac{1}{u_{j+3}^2 u_{j+4}} + \frac{1}{u_{j+2}} \left(\frac{1}{u_{j+1}} - \frac{1}{u_{j+3}} \right)^2.$$

It seems possible to arrange for both of the last two inequalities to be strict, which should allow us to indeed get $N_{i+1} < N_{j+1}$ and $M_i < M_j$!

First of all, subtracting the inequalities we obtain (note that we're taking $u_{i+2} = u_{j+2}$)

$$\left(\frac{1}{u_{i+1}} - \frac{1}{u_{i+3}}\right)^2 > \left(\frac{1}{u_{j+1}} - \frac{1}{u_{j+3}}\right)^2.$$

Since we also have

$$\frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} = \frac{1}{u_{j+1}} + \frac{1}{u_{j+3}},$$

let's take one of the smallest solutions of this system, $u_{i+1} = 2, u_{i+3} = 6$ and $u_{j+1} = u_{j+3} = 3$. Our two inequalities then greatly simplify and we see that $u_i = 10, u_{i+4} = 1, u_j = u_{j+4} = 4, u_{i+2} = u_{j+2} = 12$ indeed give a solution with strict inequalities.

We want to place these numbers as coefficients of a continued fraction that isn't unnecessarily long, so take for example $i = 1, j = 7$ and $s = 18$, and consider

$$(**) \quad \sqrt{D} = [u_0, \overline{10, 2, 12, 6, 1, 3, 4, 3, 12, 3, 4, 2, 1, 6, 12, 2, 10, 2u_0}].$$

Note that the sequence u_1, \dots, u_{s-1} is symmetric and that the coefficient $u_6 = 3$ was chosen experimentally so that everything works nicely. By Friesen's theorem [Fr], we know that there are infinitely many such squarefree integers D , so we just find one of them to get Theorem 4.

To conclude, let us sketch the argument that there are most likely infinitely many counterexamples.

For concreteness, we can continue with the example from above and take \sqrt{D} as in (**). We want to show that there are infinitely many values of u_0 such that items (1) – (5) hold as in Theorem 4.

It is straightforward to compute that there are infinitely many values of u_0 (given by a linear polynomial in a nonnegative integer variable x) and D (given by a quadratic polynomial in x) satisfying (**) – for the details of this and following computations see the Mathematica notebook at sites.google.com/site/vitakala/research/indec.

We shall later choose x so that $D \equiv 2 \pmod{4}$ is squarefree, but first one can formally compute the norms N_i of convergents. These norms are linear polynomials in x and one verifies that $N_2 = N$ is minimal (for every x). Similarly one computes that $M_7 = N(\alpha_{7,6})$ is the largest norm of a semiconvergent.

Thus for each such squarefree D , items (1), (2), (3), and (5) in Theorem 4 will be satisfied for $0 < a_0 < \frac{N}{2}$ such that $\frac{D-a_0^2}{N} = M_1$. But it could happen that this value of a is not the smallest solution of $a^2 \equiv D \pmod{N}$, as required by (4) and Conjecture 3. However, if N is prime, then this congruence has exactly two solutions $0 < a_0 < N - a_0 < N$, and hence (4) is satisfied for a_0 .

It only remains to arrange for $D \equiv 2 \pmod{4}$ (which holds when $x \equiv 2 \pmod{4}$) to be squarefree and, simultaneously, for the value of the linear polynomial $N(x)$ to be prime. It is possible to prove this as in [Er] (there is no local obstruction; in fact, in our case $D(x) = N_8(x)M_7(x)$ is the product of two coprime linear polynomials, which are also coprime with $N(x)$).

More generally, one could similarly argue for a general sequence u_1, \dots, u_{s-1} such that the error estimates in 8c) and 11b) allow one to provably determine the smallest N_i and largest M_j .

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